Solution to Assignment 5

Section 7.1

8. We have
$$\left|\int_a^b f\right| \le \int_a^b |f| \le \int_a^b M = M(b-a)\ .$$
Note that the first inequality comes from the Riemann sums after passing to limit. In the next step we integrate a constant function, see Example 2.1 in Notes 2.

11. Suppose $\lim_{n\to\infty} S(f,\dot{\mathcal{P}}_n) > \lim_{n\to\infty} S(f,\dot{\mathcal{Q}}_n)$. Then we have
$$\begin{aligned}\overline{S}(f) &= \lim_{n\to\infty} \overline{S}(f,\mathcal{P}_n) \ge \lim_{n\to\infty} S(f,\dot{\mathcal{P}}_n)\\ &> \lim_{n\to\infty} S(f,\dot{\mathcal{Q}}_n) \ge \lim_{n\to\infty} \underline{S}(f,\mathcal{Q}) = \underline{S}(f)\end{aligned}$$
$\therefore\ \overline{S}(f) \ne \underline{S}(f)$, $f \notin \mathcal{R}[a,b]$, by Integrability Criterion I.

14. (a) $\frac{1}{3}(x_{i-1}^2 + x_{i-1}x_{i-1} + x_{i-1}^2) \le q_i^2 = \frac{1}{3}(x_i^2 + x_ix_{i-1} + x_{i-1}^2) \le \frac{1}{3}(x_i^2 + x_ix_i + x_i^2)$
$\Rightarrow\quad 0 \le x_{i-1}^2 \le q_i^2 \le x_i^2 \quad\Rightarrow\quad 0 \le x_{i-1} \le q_i \le x_i.$

(b) $Q(q_i)(x_i - x_{i-1}) = \frac{1}{3}(x_i^2 + x_ix_{i-1} + x_{i-1}^2)(x_i - x_{i-1}) = \frac{1}{3}(x_i^3 - x_{i-1}^3).$

(c) Here we let $\dot{P}$ be the partition P with tags q_j. Then
$$S(Q;\dot{P}) = \sum_{i=1}^n Q(q_i)(x_i - x_{i-1}) = \frac{1}{3}\sum_{i=1}^n (x_i^3 - x_{i-1}^3) = \frac{1}{3}(b^3 - a^3)\ .$$

(d) The function $x \mapsto x^2$ is integrable by Theorem 2.8(b) being the product of the linear function $x \mapsto x$ (Example 2.4 in Notes 2). Take $\dot{P}_n$ be tagged partitions whose length tending to 0. By letting $n \to \infty$, we see from (c) and Theorem 2.6 that
$$\int_a^b Q = \frac{1}{3}(b^3 - a^3)\ .$$
Note. By choosing the tag points z_j carefully, we can use the same method to evaluate the integral of x^n for all positive powers. You are encouraged to work it out for $n = 3$. After this effort, it is easy to guess which tags to choose in the general case.

15. Let $P = \{I_j := [x_{j-1}, x_j]\}_{j=1}^n$ be a partition of f on $[a,b]$.
Clearly, $\forall\ j$, $\sup_{I_j} f = \sup_{I_j+c} g$, $\inf_{I_j} f = \inf_{I_j+c} g$. Hence $\overline{S}(f,P) = \overline{S}(g,Q)$, $\underline{S}(f,P) = \underline{S}(g,Q)$, where $Q := \{I_j + c = [x_{j-1}+c, x_j+c]\}_{j=1}^n$ is a partition of g on $[a+c, b+c]$. It is now clear that $\overline{S}(g) = \overline{S}(f)$ and $\underline{S}(g) = \underline{S}(f)$, so by the first criterion, g is integrable and
$$\int_{a+c}^{b+c} g = \overline{S}(g) = \overline{S}(f) = \int_a^b f\ .$$
Note: This property is called the translation invariance of the Riemann integral.

Section 7.2

18. If $f \equiv 0$, then result is trivial. Otherwise, since f is continuous on $[a, b]$, there exists $x_0 \in [a, b]$ s.t. $\sup f = f(x_0) > 0$. By continuity, for each small $\varepsilon > 0$, there is some δ such that $|f(x) - f(x_0)| < \varepsilon$, for all $x \in [x_0 - \delta, x_0 + \delta] \cap [a, b]$. Hence

$$\delta(f(x_0) - \varepsilon)^n < \int_{(x_0-\delta,\, x_0+\delta)\cap[a,b]} f^n \leq \int_a^b f^n \leq \int_a^b f(x_0)^n = f(x_0)^n(b-a)$$

$$\delta^{1/n}(f(x_0) - \varepsilon) < M_n = \left(\int_a^b f^n\right)^{1/n} \leq f(x_0)(b-a)^{1/n}$$

Note that $\lim_{n\to\infty} a^{1/n} = 1 \ \forall\ a > 0$. Letting $n \to \infty$, by the squeeze theorem,

$$f(x_0) - \varepsilon \leq \liminf_{n\to\infty} M_n \leq \limsup_{n\to\infty} M_n \leq f(x_0)$$

Letting $\varepsilon \to 0$, $\lim_{n\to\infty} M_n = f(x_0) = \sup\{f(x) : x \in [a, b]\}$.

19. Let P_n be the equal length partition of $[-a, 0], -a = x_0 < x_1 < \cdots < x_n = 0$, where $x_j = -a + ja/n,\ j = 0, \cdots, n$. Then we have

$$\int_{-a}^0 f = \lim_{n\to\infty} \sum_j f(x_j)\frac{a}{n} \ ,$$

see Theorem 2.6. On the other hand, $-x_j, j = 0, \cdots, n$, becomes a partition Q_n on $[0, a]$. Therefore,

$$\int_0^a f = \lim_{n\to\infty} \sum_j f(-x_j)\frac{a}{n} \ .$$

Using $f(-x) = f(x)$, we see that

$$\sum_j f(-x_j)\frac{a}{n} = \sum_j f(x_j)\frac{a}{n} \ ,$$

hence

$$\int_{-a}^0 f = \int_0^a f \ .$$

When f is odd, follow the same line but now using $\sum_j f(-x_j)\frac{a}{n} = -\sum_j f(x_j)\frac{a}{n}$ to get

$$\int_{-a}^0 f = -\int_0^a f \ .$$

Supplementary Exercises

Use the knowledge in Section 1, Notes 2.

1. (a) Find the Darboux upper and lower sums for f. Explain why the Darboux upper sum is not a Riemann sum.
   (b) Use the integrability criterion to show that f is integrable and find its integral.

Solution.

(a)
$$\begin{aligned}\overline{S}(f,P) &= \sum_{j=1}^{4} \sup_{I_j} f\ \Delta x_j \\ &= \left(\sup_{x\in[-1,-1/2]} -x\right)\left(-\frac{1}{2}-(-1)\right) + \left(\sup_{x\in[-1/2,0]} -x\right)\left(0-\left(-\frac{1}{2}\right)\right) \\ &\qquad + \left(\sup_{x\in[0,1/3]} -x+1\right)\left(\frac{1}{3}-0\right) + \left(\sup_{x\in[1/3,1]} -x+1\right)\left(1-\frac{1}{3}\right) \\ &= (1)\left(-\frac{1}{2}-(-1)\right) + \left(\frac{1}{2}\right)\left(0-\left(-\frac{1}{2}\right)\right) + (1)\left(\frac{1}{3}-0\right) + \left(\frac{2}{3}\right)\left(1-\frac{1}{3}\right) \\ &= \frac{55}{36}\end{aligned}$$

$$\begin{aligned}\underline{S}(f,P) &= \sum_{j=1}^{4} \inf_{I_j} f\ \Delta x_j \\ &= \left(\inf_{x\in[-1,-1/2]} -x\right)\left(-\frac{1}{2}-(-1)\right) + \left(\inf_{x\in[-1/2,0]} -x\right)\left(0-\left(-\frac{1}{2}\right)\right) \\ &\qquad + \left(\inf_{x\in[0,1/3]} -x+1\right)\left(\frac{1}{3}-0\right) + \left(\inf_{x\in[1/3,1]} -x+1\right)\left(1-\frac{1}{3}\right) \\ &= \left(\frac{1}{2}\right)\left(-\frac{1}{2}-(-1)\right) + 0\left(0-\left(-\frac{1}{2}\right)\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}-0\right) + 0\left(1-\frac{1}{3}\right) \\ &= \frac{17}{36}\end{aligned}$$

The Darboux upper sum is not a Riemann sum because $\sup_{[0,1/3]} f = 1$ but we can't find any tag $z \in [0,1/3]$ so that $f(z) = 1$, because of the definition of f.

(b) Take $P_n := \{x_i := -1 + i/n\}_{i=0}^{2n}$, hence $\|P_n\| \to 0$.

Then
$$\begin{aligned}\overline{S}(f) = \lim \overline{S}(f,P_n) &= \lim\left(\sum_{i=1}^{n}(-x_{i-1})\ \Delta x_i + \sum_{i=n+1}^{2n}(-x_{i-1}+1)\ \Delta x_i\right) \\ &= \lim\left(\sum_{i=1}^{2n}(-x_{i-1})\ \Delta x_i + \sum_{i=n+1}^{2n}\Delta x_i\right) \\ &= \lim\left(\sum_{i=1}^{2n}\left(1-\frac{i-1}{n}\right)\left(\frac{1}{n}\right) + \sum_{i=n+1}^{2n}\left(\frac{1}{n}\right)\right) = 2 - \lim\frac{1}{n^2}\sum_{i=1}^{2n}(i-1) + 1 \\ &= 3 - \lim\frac{1}{n^2}\frac{(0+(2n-1))2n}{2} = 3 - \lim\frac{2n-1}{n} = 3-2 = 1\end{aligned}$$

and
$$\begin{aligned}\underline{S}(f) = \lim \underline{S}(f,\mathcal{P}_n) &= \lim\left(\sum_{i=1}^{n}(-x_i)\ \Delta x_i + \sum_{i=n+1}^{2n}(-x_i+1)\ \Delta x_i\right) \\ &= \lim\left(\sum_{i=1}^{2n}(-x_i)\ \Delta x_i + \sum_{i=n+1}^{2n}\Delta x_i\right)\end{aligned}$$

$$= \lim \left(\sum_{i=1}^{2n} \left(1 - \frac{i}{n}\right) \left(\frac{1}{n}\right) + \sum_{i=n+2}^{2n} \left(\frac{1}{n}\right) \right) = 2 - \lim \frac{1}{n^2} \sum_{i=1}^{2n} i + 1$$
$$= 3 - \lim \frac{1}{n^2} \frac{(1+2n)2n}{2} = 3 - \lim \frac{1+2n}{n} = 3 - 2 = 1$$

Hence $\overline{S}(f) = 1 = \underline{S}(f)$, by integrability criterion, $f \in \mathcal{R}[-1,1]$ and $\int_{-1}^{1} f = 1$

2. Prove Cauchy criterion for integrability: f is integrable on $[a,b]$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any two tagged partitions $\dot{P}, \dot{Q}$ with length less than δ,
$$|S(f,\dot{P}) - S(f,\dot{Q})| < \varepsilon,$$
holds. (This criterion is proved in the text; pretend that it is not there.)

Solution.

$\Rightarrow$) Since $f \in R[a,b]$, $\exists\ L$ s.t. $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$,
$$|\ S(f,\dot{P}) - L\ | < \frac{\varepsilon}{2}, \quad \forall\ \|P\| < \delta.$$
For another Q, $\|Q\| < \delta$, we have a similar inequality.
$$|\ S(f,\dot{P}) - S(f,\dot{Q})\ | \le |\ S(f,\dot{P}) - L\ | + |\ S(f,\dot{Q}) - L\ | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Leftarrow$) Let $\varepsilon/2 > 0$ and choose $P = Q$ but different tags so that
$$\left| S(f,\dot{P}) - S(f,\ddot{P}) \right| < \frac{\varepsilon}{2}\ ,$$
and
$$\left| \overline{S}(f,P) - S(f,\dot{P}) \right| < \frac{\varepsilon}{4}, \quad \left| \underline{S}(f,P) - S(f,\ddot{P}) \right| < \frac{\varepsilon}{4}\ .$$
As a result,
$$\begin{aligned} \left|\overline{S}(f,P) - \underline{S}(f,P)\right| &\le \left|\overline{S}(f,P) - S(f,\dot{P})\right| + \left|S(f,\dot{P}) - S(f,\ddot{P})\right| + \left|\underline{S}(f,P) - S(f,\ddot{P})\right| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon\ . \end{aligned}$$
Therefore,
$$0 \le \overline{S}(f) - \underline{S}(f) \le \overline{S}(f,P) - \underline{S}(f,P) \le \varepsilon\ .$$
Since ε can be arbitrarily small, we must have $0 = \overline{S}(f) - \underline{S}(f)$, so f is integrable by the First Integrability Criterion.

3. Let $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = -\min\{f(x), 0\}$. Show that f_+ and f_- are both integrable when f is integrable on $[a,b]$.

Solution. Use that relation $f_+(x) = \frac{1}{2}\big(|f(x)| + f(x)\big)$, and $f_-(x) = \frac{1}{2}\big(|f(x)| - f(x)\big)$ and the integrability of $|f|$, see Theorem 2.8(d).

Alternatively, you may prove it by observing, for instance, f_+ is the composition of f and the continuous function $g(z) = z, (z \ge 0)$ and $= 0, (z < 0)$. See no 7 below.

4. Let g be differed from f by finitely many points. Show that g is integrable if f is integrable over $[a, b]$ and they have the same integral over $[a, b]$.

   Solution. For $\varepsilon > 0$, find a partition P so that
   $$\sum_P \text{osc}_j f \Delta x_j < \varepsilon/2 \ .$$
   Let $a_1, \cdots, a_m$, be the points g and f differ. They belong to at most $2m$ many subintervals of P. Hence
   $$\sum_j \text{osc}_j g \Delta x_j \leq \sum_P \text{osc}_j f \Delta x_j + 2M \times 2m \times \|P\| \ .$$
   Now we can refine the length of P so small that $4Mm\|P\| < \varepsilon/2$. Then
   $$\sum_j \text{osc}_j g \Delta x_j < \varepsilon/2 + \varepsilon/2 = \varepsilon \ ,$$
   so g is integrable. Now, let P_n with $\|P_n\| \to 0$ and choose tags equal to none of these a_j's. Then $S(g, \dot{P}_n) = S(f, \dot{P}_n)$, so their integrals are equal as $n \to \infty$.

   Alternate proof. Let $h = g - f$ so that h is equal to zero except at finitely many points. By Theorem 2.11, h is integrable and its integral is equal to 0. Therefore, $g = f + h$ is integrable and
   $$\int_a^b g = \int_a^b (f + h) = \int_a^b f + \int_a^b h = \int_a^b f \ .$$

5. Let f be non-negative and continuous on $[a, b]$. Show that $\int_a^b f = 0$ if and only if $f \equiv 0$.

   Solution. It suffices to show if f is not identically zero, then its integral is positive. Suppose there is some $x_0 \in [a, b]$ at which $f(x_0) = \alpha > 0$. By continuity, there is some small $\delta > 0$ such that $f(x) \geq \alpha/2$ for all $x \in I \equiv [x_0 - \delta, x_0 + \delta] \cap [a, b]$. Therefore,
   $$\int_a^b f \geq \int_I f \geq \int_I \frac{\alpha}{2} = \frac{\delta\alpha}{2} > 0 \ .$$

6. Let $f \in \mathcal{R}[a, b]$ and $g \in C^1[c, d]$ where $f[a, b] \subset [c, d]$. Show that the composite $g \circ f \in \mathcal{R}[a, b]$. Here C^1 means continuously differentiable.

   Solution. By MVT,
   $$g(f(x)) - g(f(y)) = g'(c)(f(x) - f(y)) \ ,$$
   where c is between $f(x)$ and $f(y)$. By assumption g' is continuous here $|g'| \leq M$ for some M. We have
   $$\sum_j \text{osc}_j g \circ f \Delta x_i \leq M \sum_j \text{osc}_j f \Delta x_j \ ,$$
   and the desired conclusion comes from the second criterion.

   Note: As a consequence of this property, the functions $|f|, f^n \ (n \geq 1), e^f, \sin f$, etc, are all integrable when f is integrable.

7. (Optional). Let $f \in \mathcal{R}[a,b]$ and $g \in C[c,d]$ where $f[a,b] \subset [c,d]$. Show that the composite $g \circ f \in \mathcal{R}[a,b]$. Hint: For $\varepsilon > 0$, fix δ_0 such that $|g(z_1) - g(z_2)| < \varepsilon$ for $|z_1 - z_2| < \delta_0$. For $\varepsilon,\ \delta_0 > 0$, there exists a partition P such that $\sum_j osc_{I_j} f \Delta x_j < \varepsilon\delta_0$. Then apply the Second Criterion.

   Solution. Given $\varepsilon > 0$, we want to find a partition P such that

$$\sum_j \text{osc}_{I_j} \Phi(f(x)) \Delta x_j < \varepsilon \ .$$

   Indeed, letting $M = \sup |f|$, Φ is uniformly continuous on $[-M, M]$. Therefore, there exists some δ such that $|\Phi(z_1) - \Phi(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta, z_1, z_2 \in [-M, M]$. For $\varepsilon_1 = \varepsilon\delta > 0$, by the Second Criterion we can find a partition P on $[a,b]$ such that

$$\sum_j \text{osc}_{I_j} f \Delta x_j < \varepsilon_1 \ .$$

   On any one of those subintervals over which osc$f$ is less than δ, we have osc $\Phi \circ f$ is less than ε. On the other hand,

$$\delta \sum_j {}' \Delta x_j \leq \sum_j {}' \text{osc}_{I_j} f \Delta x_j < \varepsilon_1 \ ,$$

   where $\sum'$ denotes the summation over those subintervals the osc of f is greater than or equal to δ. Therefore,

$$\sum_j {}' \Delta x_j \leq \frac{\varepsilon_1}{\delta} = \varepsilon \ .$$

   Putting things together, we have

$$\sum_j \text{osc } \Phi \circ f \Delta x_j = \sum_j {}' \text{osc} \Phi \circ f \Delta x_j + \sum_j {}'' \text{osc} \Phi \circ f \Delta x_j \leq C_1 \varepsilon + (b-a)\varepsilon \ ,$$

   where $\sum''$ denotes the summation over those subintervals where the osc of f is less than δ and C_1 is the oscillation of Φ over $[-M, M]$. Now we can adjust $(C_1 + (b-a))\varepsilon$ to ε.

   Note: This result is more general than the previous one.

8. Let f be a continuous function on $[a,b]$ and g a nonnegative continuous function on the same interval. Prove the mean-value theorem for integral:

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx,$$

   for some $c \in [a,b]$.

   Solution. The case is trivial when $g \equiv 0$. So we assume that $g > 0$ somewhere so that its integral is positive over $[a,b]$. Then $\int_a^b g(x)\ dx > 0$. Let $M = \sup f,\ m = \inf f$. We have

$$m \int_a^b f \leq \int_a^b fg \leq M \int_a^b f \ ,$$

   implies that

$$\int_a^b fg \Big/ \int_a^b g \in [m, M] \ .$$

As f is continuous, its range $f([a,b]) = [m, M]$. Therefore, there exists some $c \in [a,b]$ such that

$$f(c) = \int_a^b fg \Big/ \int_a^b g \ .$$

Note. Here we have used the fact that the image of an interval under a continuous function is again an interval. See Theorem 5.3.9 in [BS].